Thematic section

M
Miscellanea

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SCHEDULE OF THE SECTION
Miscellanea

- Thursday – September 7th

  14:30–15:00 Piotr Felisiak, *Generalized multiset theory and its applications*

  15:00–15:30 Piotr Mizerka, *Induction of spectral gaps for the cohomological Laplacians of* $\mathrm{SL}_n(\mathbb{Z})$ *and* $\mathrm{SAut}(F_n)$

  15:30–16:00 Leonard Mushunje, *High Dimensional Functional Data Analysis via Algebraic Geometry*

  coffee break

  16:30–17:00 Bożena Piątek, *Some types of generalized nonexpansive mappings and normal structure*

  17:00–17:30 Piotr Puchała, *A few remarks about young measures associated with bounded Borel functions*

  17:30–18:00 Adam Paszkiewicz, *The convex Peano curve does exist!*

Generalized multiset theory and its applications

Piotr Felisiak

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Abstract

The talk presents a kind of generalized multisets, existing by an axiomatic theory firstly presented by [4], and applications of these multisets in linear algebra, such as providing equivalents of classical vector operations and Gaussian elimination. Further, these generalized multisets will be called *fogs*.

A multiset is a collection of objects, in which these objects may occur more than once, *i.e.*, the collection may include several indistinguishable copies of an object. Objects contained by a multiset are called *elements* of this multiset. The number of times an element occurs in a multiset is called the *multiplicity* of the element and is a natural number. A Zermelo–Fraenkel set may be seen as a special case of a multiset, such that every element of such a multiset is with the multiplicity equal to 1. The *cardinality* of a multiset is the sum of the multiplicities of its elements. An excellent survey of literature on the multiset theories is given by W.D. Blizard in [3]. A popular introduction to the concept of multisets is [6].

Several generalizations of multisets were developed. Typically, they are based on an extension of the range of multiplicities. This talk will consider formal, axiomatic generalizations of multisets. Such generalized multisets, existing by axiomatic theories, will be further called *α-multisets*. There are not so many theories of α-multisets: in Blizard's work, [1], the multiplicity of an element is allowed to be a positive real number; in a formal theory given by him in [2], the value of a multiplicity can be positive or negative integer, what implies a possibility of negative membership; finally, Felisiak, Qin and Li in [4] propose a theory of multisets where the multiplicities are allowed to be arbitrary real numbers, that is, including negative ones, and this theory describes the fogs.

The talk explains why the fog theory may be preferred over the classical definition of multisets and their generalizations.

With fogs, it is possible to perform an equivalent of Gaussian elimination on a linear equations system (LES). Let us further call this process *foggy elimination*. An attractive feature of the foggy elimination is that

the eliminated variables (in matrix notation, zero entries) simply are not contained by the elements of the LES representation, thus these elements gradually simplify during the elimination process, while in the case of matrix notation, the matrix size remains the same for all steps of Gaussian elimination. Similarly, foggy elimination seems particularly well suited for sparse LES, since all zero coefficients in LES does not need to be represented in foggy elimination at all. Another attractive feature is that, since the writing order of elements of foggy LES representation does not matter, we do not need to carry out something analogous to the row swap operation in Gaussian elimination. One of the biggest advantages of foggy elimination is that the terms of equations do not need to be sorted by the lower indices, since as usual in fogs, the writing order of elements does not matter. This is in contrast to the Gaussian elimination, where the lower indices must correspond to column indices in the augmented matrix.

An algorithm for foggy elimination is developed, where partial pivoting is applied. The algorithm is implemented in the Python programming language.

[1] Blizard W.D., *Real-valued multisets and fuzzy sets*, Fuzzy Sets and Systems 33 (1989), no. 1, 77–97.

[2] Blizard W.D., *Negative membership*, Notre Dame Journal of Formal Logic 31 (1990), no. 3, 346–368.

[3] Blizard W.D., *The development of multiset theory*, Modern Logic 1 (1991), no. 4, 319–352.

[4] Felisiak P.A., Qin K., Li G., *Generalized multiset theory*, Fuzzy Sets and Systems 380 (2019), 104–130.

[5] Goguen J.A., *L-fuzzy sets*, Journal of Mathematical Analysis and Applications 18 (1967), no. 1, 145–174.

[6] Knuth D.E., *The art of computer programming*, 3rd ed., vol. 2: Seminumerical Algorithms, Addison Wesley 1998.

Induction of spectral gaps for the cohomological Laplacians of $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SAut}(F_n)$

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joint work with Marek Kaluba
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Abstract

In 2020 Bader and Nowak [1] showed a sufficient condition for vanishing of group cohomology with unitary coefficients. The condition involves the existence of a positive spectral gap for the first cohomological Laplacian, defined in a group ring setting. In degree one this method provides an alternative to Ozawa's [4] way to prove Kazhdan's property (T).

In this talk we focus on $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SAut}(F_n)$, the special linear group of $n \times n$ matrices over integers and the special automorphism group of the free group on n generators. We present a method to induce spectral gaps for $G_n \in \{\mathrm{SL}_n(\mathbb{Z}), \mathrm{SAut}(F_n)\}$ for cohomological Laplacians in degree one. We were inspired by the work of Kaluba, Kielak, and Nowak [2] who proved property (T) for G_n using the induction method for the Laplacian introduced by Ozawa. Using the presentations of G_n with generators being the elementary matrices for $\mathrm{SL}_n(\mathbb{Z})$ and Nielsen transvections for $\mathrm{SAut}(F_n)$, we decompose the Laplacians in degree one into summands which behave well after applying the symmetrization technique to them. This allows us to obtain a lower bound for the desired spectral gap for G_n, once we know that such a gap exists for a specific summand of the Laplacian for G_m, whenever $n \geqslant m$.

As an application of the induction technique, we were able to find explicit lower bounds for the spectral gaps for $G_n = \mathrm{SL}_n(\mathbb{Z})$ in all possible cases, that is $n \geqslant 3$. Our approach was motivated by the existence of a positive spectral gap for the Laplacian in degree one for $\mathrm{SL}_3(\mathbb{Z})$ [3] which we showed last year. Recently, we were able to find a bound for the spectral gap of the suitable part of this Laplacian which allowed us to use the induction technique and obtain the bounds for all $n \geqslant 3$. This constitutes in particular an alternative proof of property (T) for $\mathrm{SL}_n(\mathbb{Z})$.

[1] Bader U., Nowak P.W., *Group algebra criteria for vanishing of cohomology*, Journal of Functional Analysis 279 (2020), no. 11.

[2] Kaluba M., Kielak D., Nowak P.W., *On property (T) for* $\mathrm{Aut}(F_n)$ *and* $\mathrm{SL}_n(\mathbb{Z})$, Annals of Mathematics 193 (2021), no. 2, 539–562.

[3] Kaluba M., Mizerka P., Nowak P.W., *Spectral gap for the cohomological Laplacian of* $\mathrm{SL}_3(\mathbb{Z})$, arXiv:2207.02783 (2022).

[4] Ozawa N., *Noncommutative real algebraic geometry of Kazhdan's property (T)*, Journal of the Institute of Mathematics of Jussieu 15 (2016), no. 1, 85–90.

High Dimensional Functional Data Analysis via Algebraic Geometry

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Abstract

When regressing high-dimensional functional data, challenges are often encountered mainly for the in-sample than out-sample results and even worse when subjected to the curse of dimensionality. For example, on the in-sample, minimal bounds on the eigenvalues of the covariance matrix for the covariates, when using ridge regression, are not generally considered. This study aims to explore the in-sample MSPE properties of different regression methods (except ridge regression) and understand whether the eigenvalue lower bounding conditions are generally avoidable in high-dimensional Hilbert settings.

The convex Peano curve does exist!

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Abstract

We present the following main result with a number of applications and conjectures. For any convex and compact set $\mathbb{T} \subset \mathbb{R}^2$, there exists a continuous surjection $f : [0,1] \to \mathbb{T}$, such that $f(I)$ is convex for any interval $I \subset [0,1]$. If the interior $\text{int}\mathbb{T} \neq \varnothing$ then one can obtain additionally $\text{int} f(I) \neq \varnothing$ for any open interval $I \in [0,1]$.

Some types of generalized nonexpansive mappings and normal structure

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Abstract

We consider relations between a normal structure of a Banach space and the fixed point property for various classes of generalized nonexpansive mappings under additional assumptions, such as that of continuity. In this way we answer some open questions about the behaviour of such maps (see [1] and [2]).

[1] Betiuk-Pilarska A., Wiśnicki A., *On the Suzuki nonexpansive-type mappings*, Annals of Functional Analysis 4 (2013), 72–86.

[2] Fetter H., Llorens-Fuster E., *Jaggi nonexpansive mappings revisited*, Journal of Nonlinear Convex Analysis 18 (2017), 1771–1779.

A few remarks about young measures associated with bounded Borel functions

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Abstract

A probabilistic characterization of Young measures associated with bounded Borel functions is presented with some possible applications. The density of a Young measure is defined and illustrated with examples. The relations between convergence of sequences of homogeneous Young measures with densities and convergence of sequences of these densities are investigated.

[1] Fryszkowski A., *Continuous selections for a class of nonconvex multivalued maps*, Studia Mathematica 75 (1983), 163–174.

[2] Grzybowski A.Z., Puchała P., *On general characterization of Young measures associated with Borel functions*, submitted.

[3] Pedregal P., *Parametrized measures and variational principles*, Birkhäuser, Basel, Boston, Berlin, 1997.

[4] Puchała P., *An elementary method of calculating an explicit form of Young measures in some special cases*, Optimization 63 (2014), 1419–1430.

[5] Puchała P., *A simple characterization of Young measures and weak L^1 convergence of their densities*, Optimization 66 (2017), 197–203.

[6] Puchała P., *Weak convergence of the sequences of Young measures with densities*, submitted.

[7] Roubíček T., *Relaxation in optimization theory and variational calculus*, Walter de Gruyter, Berlin, New York, 1997.