Thematic section

DDS

Discrete Dynamical Systems

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SCHEDULE OF THE SECTION
Discrete Dynamical Systems

- Wednesday – September 6th

  12:00–12:30 Grzegorz Świątek, *Coefficients of the Riemann map for the Mandelbrot Set Complement*

  12:30–13:00 Xavier Jarque, *Connectivity of the basin of attraction of fixed points for some root finding algorithms*

  13:00–13:30 Piotr Oprocha, *On planar attractors and inverse limits*

- Thursday – September 7th

  12:00–12:30 Bartomeu Coll, *Asymptotic dynamic of a difference equation with a parabolic fixed point*

  12:30–13:00 Feliks Przytycki, *Iteration of rational maps on the Riemann sphere: geometric pressures and dimensions*

  14:00–14:30 Andrzej Biś, *Entropy functions for semigroup actions*

  14:30–15:00 Krzysztof Barański, *Prediction of dynamical systems*

  15:00–15:30 Marc Jorba-Cuscó, *Recurrent motion in discrete predator-prey models*

  15:30–16:00 Robert Florido-Llinàs, *Towards an atlas of wandering domains for a family of Newton maps*

  coffee break

  16:30–17:00 Krzysztof Leśniak, *Toward IFSs with non-metrizable attractors*

  17:00–17:30 Anna Jové, *Density of periodic points in boundaries of Fatou components*

  17:30–18:00 Frank Llovera, *Lorenz-like maps in classification of spike-patterns in a map-based neuron model*

- Friday – September 8th

  10:45–11:15 Michał Misiurewicz, *The Real Teapot*

  11:15–11:45 Marina Gonchenko, *Homoclinic tangencies in area-preserving maps*

  11:45–12:15 David Rojas, *Characterization of the tree cycles with minimum positive entropy for any period*

  12:15–12:45 Salvador Borrós-Cullell, *Computing regularities of invariant objects using wavelets*

  14:30–15:00 Miguel Ángel Fernandez Sanjuán, *Partial Control and Beyond: Controlling Chaotic Transients with the Safety Function*

  15:00–15:30 Josep Sardanyés, *Navigating the Unseen: transients and ghosts close to bifurcations*

  15:30–16:00 Janina Kotus, *Lyapunov exponent for non-ergodic meromorphic functions*

Prediction of dynamical systems

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joint work with Yonatan Gutman and Adam Śpiewak
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Abstract

Schroer, Sauer, Ott and Yorke conjectured in 1998 that the Takens delay embedding theorem can be improved in a probabilistic context. More precisely, their conjecture states that if μ is a natural measure for a smooth diffeomorphism of a Riemannian manifold and k is greater than the dimension of μ, then k time-delayed measurements of a one-dimensional observable are generically sufficient for a predictable reconstruction of μ-almost every initial point of the original system. This reduces by half the number of required measurements, compared to the standard (deterministic) setup. We prove the conjecture for all Lipschitz systems (also non-invertible) on compact sets with an arbitrary Borel probability measure and establish an upper bound for the decay rate of the measure of the set of points where the prediction is subpar. We also prove general time-delay prediction theorems for locally Lipschitz or Hölder systems on Borel sets in Euclidean space.

[1] Barański K., Gutman Y., Śpiewak A., *A probabilistic Takens theorem*, Nonlinearity 33 (2020), 4940–4966.

[2] Barański K., Gutman Y., Śpiewak A., *On the Shroer–Sauer–Ott–Yorke predictability conjecture for time-delay embeddings*, Communications in Mathematical Physics 391 (2022), 609–641.

[3] Barański K., Gutman Y., Śpiewak A., *Prediction of dynamical systems from time-delayed measurements with self-intersections*, arXiv:2212.13509 (2022).

Entropy functions for semigroup actions

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based on joint work with Maria Carvalho,
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Abstract

Using methods from Convex Analysis, for each generalized pressure function we define an upper semi-continuous affine entropy-like map, establish an abstract variational principle for both countably and finitely additive measures and prove that equilibrium states always exist. We study the thermodynamic formalism of continuous actions of semigroups generated by continuous self-maps or homeomorphisms of a compact metric space X. This setting comprises finitely generated semigroups, countable sofic groups and uncountable groups endowed with a reference probability measure. For each topological pressure operator associated to these actions we provide both an affine, upper semi-continuous entropy-like map, whose domain is the set of Borel probability measures on X, and a variational principle whose maximum is always attained.

Computing regularities of invariant objects using wavelets

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Abstract

This is a continuation of [3]. We want to study the family of skew products over $\mathbb{S}^1 \times \mathbb{R}$

$$\begin{pmatrix} \theta_{n+1} \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} \theta_n + \omega \mod 1 \\ F_{\sigma,\varepsilon}(\theta_n, x_n) \end{pmatrix},$$

where $\omega \in \mathbb{R}\backslash\mathbb{Q}$ and $F_{\varepsilon,\sigma} : \mathbb{R}^2 \to \mathbb{R}$ depends on two parameters ε and σ. Depending on the choice of F the system may have a Strange Non-Chaotic Attractor (SNA). We have devised a method to measure the *strangeness* of the invariant object via its *wavelet expansion.*

$$\varphi \sim a_0 + \sum_{j=0}^{\infty} \sum_{n=0}^{2^j-1} \langle \varphi, \psi_{-j,n}^{\mathrm{PER}} \rangle \psi_{-j,n}^{\mathrm{PER}}. \tag{BC}$$

where $\psi_{-j,n}^{\mathrm{PER}}$ denotes a periodized Daubechies wavelet with p vanishing moments. However, this is not an easy task, mainly due to the fact that Daubechies wavelets do not have a closed expression.

Once an expression such as in (BC) is archived, the regularity of the attractor can be approximated through Besov Spaces $\mathcal{B}^s_{\infty,\infty}$[4], since from the wavelet coefficients $\langle \varphi, \psi_{-j,n}^{\mathrm{PER}} \rangle$ one can determine the value of s such that $\varphi \in \mathcal{B}^s_{\infty,\infty}$.

Mainly we will present the results obtained by testing them on the system studied by Keller [1] and give some new results concerning the Nishikawa-Kaneko system [2].

[1] Keller G., *A note on strange nonchaotic attractors*, Fundamenta Mathematicae 151 (1996), no. 2, 139–148.

[2] Nishikawa T., Kaneko K., *Fractalization of a torus as a strange non-chaotic attractor*, Physical Review E (1996).

[3] Romero i Sánchez D., *Numerical computation of invariant objects with wavelets* (Doctoral dissertation 2015). Retrived from https://ddd.uab.cat/record/169253?ln=ca

[4] Triebel H., *Theory of Function Spaces III*, Monographs in Mathematics, Birkhäuser (2006).

Asymptotic dynamic of a difference equation with a parabolic fixed point

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Abstract

The aim of this work is the study of the asymptotic dynamical behaviour, of solutions that approach parabolic fixed points in difference equations. In one dimensional difference equations, we present the asymptotic development for positive solutions tending to the fixed point. For higher dimensions, through the study of two families of difference equations in the two and three dimensional case, we take a look at the asymptotic dynamic behaviour. To show the existence of solutions we rely on the parametrization method.

[1] Coll B., Gasull A., Prohens R., *Asymptotic Dynamics of a Difference Equation with a Parabolic Equilibrium*, Qualitative Theory of Dynamical Systems 19 (2020), no. 70.

Towards an atlas of wandering domains for a family of Newton maps

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joint work with N. Fagella

Abstract

We study the class of transcendental meromorphic functions f which are semiconjugate via the exponential to finite-type maps g in Bolsch's class [1]. Here we investigate the coexistence of wandering domains and attracting invariant basins for one-parameter families of Newton's methods f, in close relation to the forward orbit of free critical points of g, and the logarithmic lifting method for periodic Fatou components introduced by Herman [2].

[1] Bolsch A., *Iteration of meromorphic functions with countably many essential singularities*, doctoral dissertation, Technischen Universität Berlin, 1997.

[2] Herman M.R., *Are there critical points on the boundaries of singular domains?*, Communications in Mathematical Physics 99 (1995), 593–612.

Homoclinic tangencies in area-preserving maps

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Abstract

We study bifurcations in area-preserving maps with homoclinic tangencies. We consider C^r-smooth maps ($r \geqslant 3$) with a saddle fixed point whose stable and unstable invariant manifolds have a quadratic or cubic tangency at the points of some homoclinic orbit and we study bifurcations of periodic orbits near the homoclinic tangencies in perturbed area-preserving maps. Every point of these orbits can be considered as a fixed point of the so-called first return maps defined along the tangency. In the case of a quadratic homoclinic tangency, we prove the existence of cascades of generic elliptic periodic orbits for one and two parameter unfoldings. In the case of a cubic homoclinic tangency, we establish the structure of bifurcation diagram in two parameter unfoldings.

Connectivity of the basin of attraction of fixed points for some root finding algorithms

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Abstract

In this talk we discuss the connectivity of the basin of attraction of fixed points for some root finding algorithms. It is known that the Julia set of a Newton map (applied to either polynomial or entire map) is connected, and so, all Fatou components are simply connected domains (in particular the immediate basins of attraction of the fixed points associated to the zeroes of the polynomial (or entire map)). Moreover those basins are unbounded. A key argument in the proof is the non existence of weakly repelling finite fixed points. However this property is not satisfied for other root finding algorithms like Halley or Traub. We will discuss some connectivity as well as unboundedness results for those methods.

[1] Canela J., Evdoridou V., Garijo G., Jarque X., *On the basins of attraction of a one-dimensional family of root finding algorithms: from Newton to Traub*, Mathematische Zeitschrift 303 (2023), no. 55, p. 1–22.

[2] Paraschiv D. A. , *Newton-like components in the Chebyshev-Halley family of degree n polynomials*, Mediterranean Journal of Mathematics 20 (2023), no. 149.

Recurrent motion in discrete predator-prey models

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joint work with Lluís Alsedà and Tomás Lázaro

Abstract

Predator-prey models are mathematical tools used in ecology to study how predators and prey influence each other's population sizes over time. These models show that when prey populations are high, predator populations tend to increase too, leading to a decrease in prey. As prey declines, predator populations also decrease, starting the cycle again. These models help us understand the dynamics of ecosystems but may not capture all complexities in real-life interactions.

In this work we consider the following discrete dynamical system:

$$\begin{cases} \bar{x} = \mu x(1 - x - y), \\ \bar{y} = \beta xy. \end{cases}$$

Here, the preys x grows logistically with an intrinsic reproduction rate μ and the predators y increase their population numbers at rate β. Although very simple, this model displays very rich dynamics (see [1]). We discuss recurrent motion (periodic, quasi-periodic and chaotic orbits) and identify trajectories which are not stable but behave as they were for some amount of time.

[1] Vidiella B., Lázaro J.T., Alsedà L., Sardanyés J., *On Dynamics and Invariant Sets in Predator-Prey Maps*, Dynamical Systems Theory, IntechOpen, Rijeka 2019.

Density of periodic points in boundaries of Fatou components

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joint work with N. Fagella

Abstract

This talk concerns holomorphic dynamics, and the study of Fatou and Julia sets. In particular, we will address the problem of finding periodic points in the boundary of attracting and parabolic basins, and some types of Baker domains. For rational maps, F. Przytycki and A. Zdunik proved that periodic points are always dense in the boundary of attracting or parabolic basins. New ideas and techniques to work with transcendental functions will be provided.

Lyapunov exponent for non-ergodic meromorphic functions

Janina Kotus

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Abstract

Levin, Przytycki and Shen proved for a polynomial map $f_c(z) = z^d + c$, $d \geqslant 2$ and $c \in \mathbb{C}$ with Julia set $J(f_c)$ of positive measure that for a.e. $z \in J(f_c)$ the Lyapunov exponent $\chi_s(z) = 0$. Dobbs proved that transcendental entire this is result is not true. We will show that it does not occur also for transcendental meromorphic functions with poles.

Toward IFSs with non-metrizable attractors

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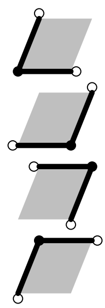
joint work with Magdalena Nowak

Abstract

We are going to construct some iterated function systems (IFS) with non-metrizable attractors, like a split square; see the picture on the right.



[1] Barnsley M.F., Leśniak K., Rypka M., *Chaos game for IFSs on topological spaces*, Journal of Mathematical Analysis and Applications 435 (2016), no. 2, 1458–1466.

[2] Leśniak K., Nowak M., *Split square and split carpet as examples of non-metrizable IFS attractors*, arXiv:2208.14253 (2022).

Lorenz-like maps in classification of spike-patterns in a map-based neuron model

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joint work with Piotr Bartłomiejczyk
and Justyna Signerska-Rynkowska

Abstract

We study the well-known map-based model of neuronal dynamics introduced in 2007 by Courbage, Nekorkin and Vdovin, important due to various medical applications. We also review and extend some of the existing results concerning β-transformations and (expanding) Lorenz mappings. Then we apply them for deducing important properties of the spike-trains generated by the CNV model and explain their implications for the neuron behaviour. In particular, using recent theorems of rotation theory for Lorenz-like maps, we provide a classification of periodic spiking patterns in this model.

[1] Bartlomiejczyk P., Llovera F., Signerska-Rynkowska J., *Spike patterns and chaos in a map-based neuron model*, International Journalof Applied Mathematics and Computer Science 33 (2023), no. 3.

The Real Teapot

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joint work with Lluis Alseda, Jozef Bobok, Lubomir Snoha

Abstract

In his last paper, William Thurston defined the Master Teapot as the closure of the set of pairs (z, s), where s is the slope of a tentmap T_s with the turning point periodic, and a complex number z isa Galois conjugate of s. In this case $1/z$ is a zero of the kneading determinant of T_s. We remove the restriction that the turning point is periodic, and sometimes look beyond tent maps.

However, we restrict our attention to zeros $x = 1/z$ in the real interval $(0, 1)$. By the results of Milnor and Thurston, the kneading determinant has such a zero if and only if the map has positive topological entropy. We show that the first (smallest) zero is simple, but among other zeros there may be multiple ones. We describe a class of unimodal maps, so-called R-even ones, whose kneading determinant has only one zero in $(0, 1)$. In contrast with this, we show that generic mixing tent maps have kneading determinants with infinitely many zeros in $(0, 1)$. We prove that the second zero in $(0, 1)$ of the kneading determinant of a unimodal map, provided it exists, is always larger than or equal to $\sqrt[3]{1/2}$ and if the kneading sequence begins with $RL^N R$, $N \geqslant 2$, then the best lower bound for the second zero is in fact $\sqrt[N+1]{1/2}$. We also investigate (partially numerically) the shape of the *Real Teapot*, consisting of the pairs (s, x), where x in $(0, 1)$ is a zero of the kneading determinant of T_s, and $s \in (1, 2]$.

On planar attractors and inverse limits

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joint work with Jernej Činč

Abstract

A very useful technique called BBM (Brown-Barge-Martin), incorporates inverse limits and natural extensions of the underlying bonding maps to embed attractors in manifolds. The original idea goes back to the paper of Barge and Martin, where the authors constructed strange attractors from a wide class of inverse limits. One of the crucial steps for this technique to work is the usage of Brown's approximation theorem. Recently, this technique was extended to produce a parameterized family of strange attractors. In this talk we will present a few possible applications of BBM technique in construction of concrete examples.

Iteration of rational maps on the Riemann sphere: geometric pressures and dimensions

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Abstract

I will pose some questions on

1. Geometric pressure via periodic orbits.
2. Dimension of Julia sets as the first zero of the geometric pressure.
3. Geometric coding trees,

Characterization of the tree cycles with minimum positive entropy for any period

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joint work with Francesc Mañosas and David Juher

Abstract

The notion of *pattern* plays a central role in the theory of topological and combinatorial dynamics. Consider a family $\mathcal{X}$ of topological spaces (for instance the family of either all closed intervals of the real line or all trees or all graphs or compact surfaces, etc) and the family $\mathcal{F}_{\mathcal{X}}$ of all maps $\{f : X \to X : X \in \mathcal{X}\}$ satisfying a given restriction (continuous maps, homeomorphisms, etc). Given a map $f : X \to X$ in $\mathcal{F}_{\mathcal{X}}$ which is known to have a finite invariant set P, the *pattern of* P *in* $\mathcal{F}_{\mathcal{X}}$ is the equivalence class $\mathcal{P}$ of all maps $g : Y \to Y$ in $\mathcal{F}_{\mathcal{X}}$ having an invariant set $Q \subset Y$ that, at a combinatorial level, behaves like P. That is, the relative positions of the points of Q inside Y are the same as the relative positions of P inside X, and the way these positions are permuted under the action of g coincides with the way f acts on the points of P. In this case, it is said that every map g in the class *exhibits* the pattern $\mathcal{P}$. If in particular P is a periodic orbit of f, the pattern $\mathcal{P}$ is said to be *periodic*.

In this talk we deal with patterns of invariant sets of continuous maps defined on trees (simply connected graphs). We consider, for any integer $n \geqslant 3$, the set Pos_n of all n-periodic tree patterns with positive topological entropy. We explicitly construct an n-periodic tree pattern $\mathcal{Q}_n$ whose entropy is minimum in Pos_n.

Partial Control and Beyond: Controlling Chaotic Transients with the Safety Function

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joint work with G. Alfaro and R. Capeáns from URJC, Spain

Abstract

A new control algorithm based on the partial control method has been developed. The general situation we are considering is an orbit starting in a certain phase space region Q having a chaotic transient behavior affected by a bounded noise, so that the orbit will definitely escape from Q in an unpredictable number of iterations. Thus, the goal of the algorithm is to control in a predictable manner when to escape. While partial control has been used as a way to avoid escapes, here we want to adapt it to force the escape in a controlled manner. We have introduced new tools such as escape functions and escape sets that once computed makes the control of the orbit straightforward. The partial control method aims to avoid the escape of orbits from a phase space region Q where the transient chaotic dynamics takes place. The technique is based on finding a special subset of Q called the safe set. The chaotic orbit can be sustained in the safe set with a minimum amount of control. We have developed a control strategy to gradually lead any chaotic orbit in Q to the safe set by using the safety function. With the technique proposed here, the safe set can be converted into a global attractor of Q. In addition, we deal with the Hénon and the Lozi maps for a choice of parameters where they show transient chaos, and we compute their safety functions showing the strong dependence of the safety function with the strength of the bounded noise affecting the maps, drastically impacting the controlled orbits.

[1] Alfaro G., Capeáns R., Sanjuán M.A.F., *Forcing the escape: Partial control of escaping orbits from a transient chaotic region*, Nonlinear Dynamics 104 (2021), 1603–1612.

[2] Capeáns R., Sanjuán M.A.F., *Beyond partial control: Controlling chaotic transients with the safety function*, Nonlinear Dynamics 107 (2022), 2903–2910.

[3] Capeáns R., Sanjuán M.A.F., *Controlling chaotic transients in the Hénon and the Lozi map with the safety function*, Journal of Difference Equations and Applications (2023).

[4] Sabuco J., Sanjuán M.A.F., Yorke J.A., *Dynamics of Partial Control*, Chaos 22 (2012).

Navigating the Unseen: transients and ghosts close to bifurcations

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Abstract

It is known that transients towards attractors become longer close to bifurcations. This phenomenon, which is found in local and global bifurcations, can have deep implications in multitude of dynamical systems such as ecosystems. Such implications range from delays in population collapses or in difficulties in the recovery of ecosystems with multiple stable states. In this talk we will focus on so-called ghost transients which typically appear after saddle-node bifurcations. We will focus on the properties of these transients investigated in discrete-time and discrete-time-space ecological models. Other examples of ghost manifolds will be introduced.

[1] Canela J., Fagella N., Alsedà Ll., Sardanyés J., *Dynamical mechanism behind ghosts unveiled in a map complexification*, Chaos, Solitons & Fractals 156 (2022).

[2] Dai L., Vorselen D., Korolev K.S., Gore J., *Generic Indicators for Loss of Resilience Before a Tipping Point Leading to Population Collapse*, Science 336 (2012), no. 6085, 1175–1177.

[3] Duarte J., Januário C., Martins N., Sardanyés J., *On chaos, transient chaos and ghosts in single population models with Allee effects.* Nonlinear Analysis Series B 13 (2012), no. 4, 1647–1661.

[4] Sardanyés J., Solé R., *Bifurcations and phase transitions in spatially extended two-member hypercycles*, Journal of Theoretical Biology 234 (2006), no. 4, 468–482.

[5] Trickey S.T., Virgin L.N., *Bottlenecking phenomenon near a saddle-node remnant in a Duffing oscillator*, Physics Letters A 248 (1998), no. 2, 185–190.

Coefficients of the Riemann map for the Mandelbrot Set Complement

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joint with Grzegorz Siudem of *Warsaw Tech.*

Abstract

We propose a numerical algorithm for the approximate computation of the coefficients of the standardized Riemann map for the complement of the Mandelbrot set. Let $f_c(z) = z^2 + c$ be the standard representation of the quadratic family and $\mathcal{M}$ the Mandelbrot set and $\Psi : \overline{\mathbb{C}}\setminus\overline{\mathbb{D}(0,1)} \to \overline{\mathbb{C}}\setminus\mathcal{M}$ is normalized Riemann map. We start with the Fourier transform of the normalized Riemann map given with the formula $\Psi^{-1}(c) = \lim_{n\to\infty} \sqrt[2^n]{f_c^n(c)}$. The application of Fourier transform reduces the problem to the simple relation, solvable with Newton's method. Replacing the full Fourier series equivalent to the power series expansion by a discrete Fourier transform introduces a certain error, but it is easy to estimate and control. The method has enabled us to compute 2^n coefficients of the Riemann map in $O\left(n2^n\right)$ operations and obtain coefficients up to $n = 26$ on standard hardware and high accuracy. Collected data suggest that the coefficients decay fairly quickly but insufficiently to make the sequence summable and certain other conjectures were validated or posed. The algorithm is easily parallelizable in the most computationally demanding part using the Newton's method and, less easily, using a fast parallel implementation of the Fourier transform.

[1] Bittner D., Cheong L. et al., *New aproximations for the area of the Mandelbrot set*, Involve 10 (2017).

[2] Carleson L., Jones P., *On coefficient problems for univalent functions and conformal dimension*, Duke Mathematical Journal 66 (1992), 169–206.

[3] Ewing J., Schober G., *The area of the Mandelbrot set*, Numerische Mathematik 61 (1992), 59–72.

[4] Graczyk J., Smirnov S., *Collet, Eckmann, & Hölder*, Inventiones Mathematicae 133 (1998), 69–96.

[5] Jungreis I., *The uniformization of the complement of the Mandelbrot set*, Duke Mathematical Journal 52 (1985), 935–938.

[6] Levin G., *Theory of iterations of polynomial families in the complex plane*, Journal of Soviet Mathematics 5 (1990), 3512–3522.