Thematic section:

DDE
Dynamics of Differential Equations

ORGANIZERS:

Maciej Capiński (Akademia Górniczo-Hutnicza, Kraków)

Amadeu Delshams (Universitat Politecnica de Catalunya, Barcelona)

Marcel Guardia (Universitat Politecnica de Catalunya, Barcelona)

Piotr Zgliczyński (Uniwersytet Jagielloński, Kraków)

SCHEDULE OF THE SECTION
Dynamics of Differential Equations

- Monday – September 4th

  16:00–16:30 Andrew Clarke, *Why are inner planets not inclined?*

  16:30–17:00 Óscar Rodríguez, *Dynamical study of Hilda asteroids through quasi-periodic solutions*

  coffee break

  17:30–18:00 Aleksander Pasiut, *Oscillatory orbits to collision in the planar circular restricted three body problem*

  18:00–18:30 Piotr Zgliczyński, *Shadowing of non-transversal heteroclinic chains*

  18:30–19:00 Amadeu Delshams, *Polynomial normal forms for ODEs preserving some dynamical structures*

- Tuesday – September 5th

  14:30–15:00 José Lamas Rodríguez, *Parabolic ejection & collision orbits for the restricted planar circular three body problem*

  15:00–15:30 Rodrigo G. Schaefer, *Arnold Diffusion via Scattering maps: A geometrical mechanism to detect global instability*

  15:30–16:00 Maciej Capiński, *Characterising blenders via covering relations and cone conditions*

  coffee break

  16:30–17:00 Robert Szczelina, *A geometric method for computer assisted proofs in delay differential equations*

  17:00–17:30 Pau Martín, *Chaotic scattering of He atoms off a Cu surface with corrugated Morse potential*

  17:30–18:00 Małgorzata Moczurad, *Central configurations for $N + k$ body problem*

- Thursday – September 7th

  14:00–14:30 Chen Xuan, *The maths of a photo induced hydrogel swimming robot: nonsmooth forcing dynamics*

  14:30–15:00 Piotr Kalita, *On unbounded attractors for dynamical systems*

  15:00–15:30 Juan Garcia Fuentes, *Geometrical description of forwards attractors for non-autonomous Lotka-Volterra systems*

  15:30–16:00 Maria Przybylska, *TBA*

  coffee break

  16:30–17:00 Andrzej Maciejewski, *TBA*

Why are inner planets not inclined?

Andrew Clarke

Universitat Politècnica de Catalunya
email: andrew.clarke@ub.edu

and Jacques Fejoz, *Université Paris Dauphine, Observatoire de Paris*
and Marcel Guardia, *Universitat de Barcelona, Centre de Recerca Matemàtica*

Abstract

Consider the Newtonian 4-body problem, in the regime where 3 bodies (the planets) revolve on near-elliptical orbits around the other body (the sun). A long-held belief, culminating in the XVIII century in the first stability theorem of Laplace and Lagrange, is that the semimajor axes are stable. Assuming the initial conditions of the semimajor axes are of different orders, and that there is a large mutual inclination between planets 1 and 2, we prove that there are orbits of the 4-body problem where the semimajor axis of planet 3 can follow any itinerary, with arbitrary precision. In addition, along such orbits, we can make the normalised angular momentum vector of planet 2 follow any itinerary, as well as the eccentricity of the orbital ellipse of planet 2, again with arbitrary precision. For example, planet 2 may flip from prograde to retrograde nearly- horizontal revolutions. These orbits constitute a counterexample to the first stability theorem of Laplace and Lagrange. Moreover, as a consequence of the proof, the non-recurrent set of any finite-order secular normal form accumulates on circular motions, thus proving a weak form of a celebrated conjecture of Herman.

Polynomial normal forms for ODEs preserving some dynamical structures

Amadeu Delshams

UPC and CRM
Barcelona
email: Amadeu.Delshams@upc.edu

joint work with Piotr Zgliczyński

Abstract

We prove the polynomial normal form theorem for flows that preserve some dynamical structures. These new structures are the preservation of some invariant subspaces and the block diagonal structure of the variational equations along them, which are important for the shadowing of nontransverse heteroclinic chains.

Geometrical description of forwards attractors for non-autonomous Lotka-Volterra systems

Juan Garcia Fuentes

Universidad de Sevilla
Ecuaciones Diferenciales y Análisis Numérico
email: jgfuentes@us.es

joint work with José Antonio Langa, Piotr Kalita and Antonio Suárez, fully developed in [3]

Abstract

The autonomous Lotka-Volterra model is used to study the evolution of species population from an ecosystem, and the full characterization of the asymptotic behavior of their solutions is well known. In particular, it is possible to give conditions on the coefficients which guarantee the existence of a globally asymptotically stable equilibrium point, in which all the species are present (known as a permanence solution), or, on the contrary, conditions to obtain a stable equilibrium point which possess one or more species extinct. Futhermore, one can construct the full structure of the global attractor associated through the heteroclinics conections between equilibrium points [1]. Passing these results to the non-autonomous situation, i.e. when the parameters are depending on time, is non-trivial, and in general, it is rare to get time dependent invariant compact attracting sets when time goes to $+\infty$. Based on the works of Lazer and Ahmad [1], [2], and Redheffer [4], we consider a non-autonomous Lotka-Volterra system where the species cooperate between them. We give sufficient conditions to obtain the existence of a globally asymptotically stable solutions, in both the permanence and extinction situations. Once obtained the stable solution, we obtain the exact geometrical structure of the forwards non-autonomous attractor by constructing the heteroclinic connections between the globally stable solution and the semistables ones.

[1] Ahmad S., Lazer A., *On the nonautonomous N-competing species Problems*, Applicable Analysis 57 (1995), 309–323.

[2] Ahmad S., Lazer A., *Necessary and sufficient average growth in a Lotka–Volterra system*, Nonlinear Analysis 34 (1998), 191–228.

[3] Garcia-Fuentes J., Langa J.A., Kalita P., Suárez A., *Characterization of attractors for non-autonomous Lotka-Volterra cooperative systems*, arXiv:2301.04955 (2023).

[4] Redheffer R., *Nonautonomous Lotka–Volterra systems*, International Journal of Differential Equations 127 (1996), 519–541.

[5] Takeuchi Y., *Global asymptotic dynamical properties of Lotka-Volterra systems*, World Scientific Publishing, 1996.

On unbounded attractors for dynamical systems

Piotr Kalita

Jagiellonian University, Kraków
Faculty of Mathematics and Computer Science
email: piotr.kalita@uj.edu.pl

Abstract

If the semigroup is slowly non-dissipative, i.e., its trajectories can diverge to infinity as time tends to infinity, one still can study its dynamics via the approach by the unbounded attractors - the counterpart of the classical notion of global attractors. We continue the development of this theory started by Chepyzhov and Goritskii in [4] and extended recently in [1, 2, 3]. We provide the abstract results on the unbouded attractor existence in slowly non-dissipative setting for autonomous and non-autonomous situation. The abstract theory that we develop is illustrated by the analysis of the problem governed by the equation

$$u_t = Au + f(u),$$

as well as its non-autonomous counterpart. In particular, using the notion of inertial manifold and graph transform approach and the Lyapunov-Perron approach, we provide the criteria under which the unbounded attractor coincides with the graph of the Lipschitz function, or becomes close to the graph of the Lipschitz function for large argument. At the same time, we derive the new non-autonomous spectral gap-type conditions which guarantee the existence of non-autonomous inertial manifolds in the unbounded attractor setting.

[1] Banaśkiewicz J., Carvalho A.N., Garcia-Fuentes J., Kalita P., *Autonomous and non-autonomous unbounded attractors in evolutionary problems*, Journal of Dynamics and Differential Equations (2022), doi: 10.1007/s10884-022-10239-x.

[2] Bortolan M.C., Fernandes J., *Sufficient conditions for the existence and uniqueness of maximal attractors for autonomous and nonautonomous dynamical systems*, Journal of Dynamics and Differential Equations (2022), doi: 10.1007/s10884-022-10220-8.

[3] Bortolan M.C., Fernandes J., Kalita P., *On unbounded attractors in dynamical systems*, in preparation.

[4] Chepyzhov V.V., Goritskii A.Yu., *Unbounded attractors of evolution equations*, Advances in Soviet Mathematics 10, 85—128, American Mathematical Society, 1992.

Parabolic ejection & collision orbits for the restricted planar circular three body problem

José Lamas Rodríguez

Universitat Politècnica de Catalunya
Departament de Matemàtiques
email: jose.lamas.rodriguez@upc.edu

and Marcel Guàrdia, *Departament de Matemàtiques, Universitat de Barcelona; Centre de Recerca Matemàtica, Campus de Bellaterra, Edifici C.*
email: guardia@ub.edu
and Tere M-Seara, *Departament de Matemàtiques, Universitat Politècnica de Catalunya; Centre de Recerca Matemàtica, Campus de Bellaterra, Edifici C.*
email: tere.m-seara@upc.edu

Abstract

We consider the restricted planar circular three body problem (RPC3BP), which describes the motion of a massless body under the attraction of other two bodies, the primaries, which describe circular orbits around their common center of mass located at the origin.

In a suitable system of coordinates, this system is Hamiltonian with two degrees of freedom, whose conserved energy is usually called the Jacobi constant. In such system, we are interested in solutions of the RPC3BP called *ejection-collision orbits*, i.e., solutions that depart from the big primary at some time t_0 and collide with it at some instant t_1.

In this talk I will explain how to construct arbitrarily large ejection-collision orbits for small values of the mass ratio. To this end, we show that, for small values of the mass ratio and the Jacobi constant, there exist transverse intersections between the stable (unstable) manifold of *infinity* and the unstable (stable) manifold of collision.

Close to such transverse intersections, we prove the existence of a sequence of ejection-collision orbits that travel arbitrarily far away. Moreover, using a similar argument, we prove the existence of a sequence of forward and backward periodic parabolic orbits that travel close to collision too. Finally, we also prove the existence of periodic orbits that travel close to collision and arbitrarily far away.

Chaotic scattering of He atoms off a Cu surface with corrugated Morse potential

Pau Martín

Universitat Politècnica de Barcelona
Centre de Recerca Matemàtica
email: p.martin@upc.edu

and F. Borondo, *Universidad Politécnica de Madrid*
and E. Fontich, *Universitat de Barcelona*

Abstract

We consider the motion of an helium atom bouncing off a copper surface. We assume that the He atom motion takes place on a plane. We denote by $(x, z) \in \mathbb{R}^2$ the position of the He atom, where x and z are the horizontal and vertical displacements, respectively. Let (p_x, p_z) be the conjugate momenta. We assume that the interaction of the He atom with the copper surface is modeled by a corrugated Morse potential, where the corrugation represents the presence of the copper atoms in the surface. More concretely, we consider the Hamiltonian

$$H_{\mathrm{CM}}(x, z, p_x, p_z) = \frac{1}{2m}(p_x^2 + p_z^2) + V_{\mathrm{M}}(z) + V_{\mathrm{C}}\left(\frac{2\pi x}{a}, z\right), \qquad \text{(M1)}$$

where

$$\begin{aligned} V_{\mathrm{M}}(z) &= De^{-\alpha z}(e^{-\alpha z} - 2), \\ V_{\mathrm{C}}(\theta, z) &= De^{-2\alpha z} V(\theta), \\ V(\theta) &= \sum_{n \geqslant 1} r_n \cos(n\theta) + s_n \sin(n\theta). \end{aligned}$$

The coefficients r_n and s_n are determined experimentally. Their values are

$$r_1 = 0.06, \qquad r_2 = 0.008, \qquad r_n = 0, \quad n \geqslant 3, \qquad s_n = 0, \quad n \geqslant 1,$$

$D = 6.35$ meV, $a = 3.6$ Å, $\alpha = 1.05$ Å^{-1}. In particular, $r_1 > 0$. In the present paper, our only requirement on V is that it is an analytic even function.

The purpose of this paper is twofold. On the one hand, we prove presence of chaos in some parts of the phase space of (M1). Here, the

notion of chaos is the one introduced by Smale and it is based on the presence of a *Smale horseshoe* and a conjugation with the shift on a space of symbols. It is worth to remark that the set where chaos takes place is a hyperbolic set.

On the other, we prove the existence of *oscillatory orbits*, that is, solutions $(x(t), z(t))$ of (M1) with the property that $\limsup_t z(t) = \infty$ and $\liminf_t z(t) < \infty$, that is, solutions such that go higher and higher but always go back again to a finite distance of the Cu surface.

Central configurations for $N + k$ body problem

Małgorzata Moczurad

Jagiellonian University
email: malgorzata.moczurad@uj.edu.pl

joint work with Piotr Zgliczyński

Abstract

We study the problem of planar central configurations with N heavy bodies and k bodies with arbitrary small masses.

We derive the equation describing the limit of light masses going to zero, which can be seen as the equation for central configurations in the anisotropic plane. Using computer rigorous computations we compute

all central configurations for $N = 2$ and $k = 3, 4$ and for the derived limit problem. We show that the results are consistent.

Oscillatory orbits to collision in the planar circular restricted three body problem

Aleksander Pasiut

AGH University of Science and Technololgy
Faculty of Applied Mathematics
email: pasiut@agh.edu.pl

Abstract

In our work, we present the computer assisted proof of the existence of infinite number of periodic orbits of arbitrarily long period that approach one of the primary masses of the system arbitrarily close. In order to achieve our goal we make use of topological tools (such as covering relations) and we perform rigorous interval computations with CAPD package. The proof is based on performing the Levi-Civita regularization, showing the existence of the particular orbits with numerical and topological tools and proving that the orbits remain valid when switching back to the non-regularized system.

[1] Alefeld G., Mayer G., *Interval analysis: theory and applications*, Journal of Computational and Applied Mathematics 121 (2000), no. 1, 421–464.

[2] Kapela T., Mrozek M., Wilczak D., Zgliczyński P., *CAPD::DynSys: A flexible C++ toolbox for rigorous numerical analysis of dynamical systems*, Communications in Nonlinear Science and Numerical Simulation 101 (2021).

[3] Levi-Civita T., *Sur la résolution qualitative du problème restreint des trois corps*, Acta Mathematica 30 (1906), 305–327.

[4] Zgliczyński P., Gidea M., *Covering relations for multidimensional dynamical systems*, Journal of Differential Equations 202 (2004), no. 1, 32–58.

Dynamical study of Hilda asteroids through quasi-periodic solutions

Óscar Rodríguez

Universitat Politècnica de Catalunya
Dept. Matemàtiques
email: oscar.rodriguez@upc.edu

and Àngel Jorba, *Universitat de Barcelona,*
Dept. Matemàtiques i Informàtica
and Begoña Nicolás, *Universidade de Santiago de Compostela,*
Dept. Matemáticas

Abstract

The Hilda family of asteroids is a group of more than 5000 asteroids located beyond the main asteroid belt of our Solar System, but within Jupiter's orbit. They are known to have mean motion in a 3 : 2 orbital resonance with Jupiter and to describe orbits that seem to successively approach three Lagrangian Points, L_3, L_4 and L_5, of the Sun-Jupiter system.

Our aim in this work is to analyse Hilda's behaviour from a dynamical systems ap- proach, by studying their orbits within Sun-Jupiter Circular Restricted Three Body Prob- lem (CRTBP) and Elliptical Restricted Three Body Problem (ERTBP), both in the planar case. The reason for studying both models is to analyse the level of importance of Jupiter eccentricity in this particular application.

Our analysis starts by selecting those asteroids in the JPL database with orbital elements of the Hilda category, although focusing on those with low inclination. The database provides the coordinates of the asteroids in an inertial ecliptical reference frame, with the origin set at the solar system center of mass. Then, a change of coordinates for the ephemeris of these asteroids is needed in order to have them in the CRTBP or ERTBP Sun-Jupiter systems. This (non trivial) change of coordinates is defined through the instantaneous orbital elements of the Sun and Jupiter and it will be detailed in the presentation.

Once we have the coordinates of the Hilda asteroids in our mathematical models, we are in position of computing numerically the periodic and quasi-periodic orbits that are assumed to be responsible of their motion. In order to compute these quasi-periodic orbits (also known as invariant

tori), we make use of temporal or spatial Poincarè sections. Some comparisons have been performed to identify the most convenient and effective strategy when analysing these invariant objects and their stability.

Some results will be presented for both the circular and the elliptical restricted three body problems.

Arnold Diffusion via Scattering maps: A geometrical mechanism to detect global instability

Rodrigo G. Schaefer

Jagiellonian University
Faculty of Mathematics and Computer Science
email: rodrigo.goncalves.schaefer@uj.edu.pl

joint work with Amadeu Delshams (UPC) and Albert Granados

Abstract

We proved, in [1, 2], that for any non-trivial perturbation depending on any two independent harmonics of a pendulum and a rotor there is global instability, also called Arnold diffusion. The proof is based on the geometrical method and relies on the concrete computation of several scattering maps. A complete description of the different kinds of scattering maps taking place as well as the existence of piecewise smooth global scattering maps is also provided. Similar results apply for any non-trivial perturbation depending on any three independent harmonics of a pendulum and a 2 d.o.f rotor [3].

[1] Delshams A., Schaefer R.G., *Arnold diffusion for a complete family of perturbations*, Regular and Chaotic Dynamics 22 (2017), no. 1, 78–108.

[2] Delshams A., Schaefer R.G., *Arnold diffusion for a complete family of perturbations with two independent harmonics*, Discrete and Continuous Dynamical Systems 38 (2018), no. 12.

[3] Delshams A., Granados A., Schaefer R.G., *Arnold Diffusion for a Hamiltonian system with 3+1/2 degrees of freedom*, in preparation (2023).

A geometric method for computer assisted proofs in delay differential equations

Robert Szczelina

Jagiellonian University
Faculty of Mathematics and Computer Science
email: robert.szczelina@uj.edu.pl

Abstract

A covering relation is a tool to express a concept that a given map $f : \mathbb{R}^n \to \mathbb{R}^n$ stretches in a proper fashion one set over another. Covering relations can be used to obtain coding for the orbits of the system, which is generally referred to as symbolic dynamics. Due to their geometric nature and open conditions, covering relations can be rigorously checked with computer assistance. We will present one possible exten- sion of the finite-dimensional covering relations to infinite dimensional systems, using compactness of the map.

A recently developed high-order Lohner-type rigorous algorithm [1] can be used to get enclosures of solutions to systems of Delay Differential Equations (DDEs) of quality good enough for various computer assisted proofs. We apply this method to verify covering relations for some Poincaré maps in the (subspace) of the phase space $C^0\left([-\tau, 0], \mathbb{R}^d\right)$ of DDEs to prove several unstable periodic orbits to Mackey-Glass equation in the chaotic regime of parameters, and the persistence of symbolic dynamics (semiconjugacy to a subshift on two symbols) in a chaotic ODE perturbed with a delayed term.

[1] Szczelina R., Zgliczyński P., *High-order Lohner-type algorithm for rigorous computation of Poincaré maps in systems of Delay Differential Equations with several delays*, Foundations of Computational Mathematics, in press (2023).

The maths of a photo induced hydrogel swimming robot: nonsmooth forcing dynamics

Chen Xuan

Department of Foundational Mathematics
Xi'an Jiaotong-Liverpool University
and
Leonard S. Ornstein Laboratory
Utrecht University
email: Chen.xuan@xjtlu.edu.cn

Abstract

Certain hydrogel is light sensitive, since light provides heat and hydrogel shrinks under high temperature thus making the hydrogel photosensitive. When light shines on the hydrogel, the hydrogel shrinks and as light propagates through the hydrogel shrinks less thus bending towards the light. This hydrogel beam undergoes a vibration, governed by a wave PDE with a second derivative in time and 4th derivative in space coupled with a diffusion equation, a result of a competition between the elasticity of the beam and the photo induced bending. This photo induced vibration enables the hydrogel beam to swim in water. Based on dimensional energy analysis, we determine the stable vibration amplitude and construct phase diagrams for the increase and decrease of the oscillation amplitude, which are further confirmed experimentally. It is found that resonance can occur and damping plays an important role in determining the conditions for resonance. A mass-spring-damper ODE system subjected to a displacement dependent excitation force is developed to investigate the features in generalized self-excited oscillating systems. The prototypical PDEs can be well understood by the above simplified ODE model. This work lays a solid foundation for understanding self-excited oscillation and provides design guidelines for self-sustainable soft robots. It also puts forth another interesting question of whether chaos is involved in future work.

Shadowing of non-transversal heteroclinic chains

Piotr Zgliczyński

Jagiellonian University
Faculty of Mathematics and Computer Science
email: umzglicz@cyf-kr.edu.pl

joint work with Amadeu Delshams

Abstract

We discuss a geometric method for the shadowing of nontransversal chain of heteroclinic connections based on the idea of dropping dimensions.

In our picture we think of evolving a disk of dimension k along a heteroclinic chain and when a given transition is not transversal, then we 'drop' one or more dimensions of our disk, i.e., we select a subdisk of lower dimension "parallel to expanding directions in future transitions". After at most k transitions, our disk is a single point and we cannot continue further. We will refer to this phenomenon as the *dropping dimensions* mechanism.

We illustrate this new mechanism for a generalization of toy model systems introduced by Colliander and all, Guardia and Kaloshin, Guardia-Hauss-Processi in the study of the energy transfer to high frequencies in the cubic defocusing nonlinear Schrödinger equation.