Thematic section

AST
Applications of Set Theory

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SCHEDULE OF THE SECTION
Applications of Set Theory

- Monday – September 4th

  16:00–17:00 David Asperó, *Forcing axioms beyond* $H(\omega_2)$

  coffee break

  17:30–18:30 Jorge Lopez-Abad, *On Pelczynski universal space: double Fraïssé classes*

  18:30–19:00 Arturo Martínez-Celis, *Ultraproducts and Michael spaces*

- Tuesday – September 5th

  14:30–15:30 Piotr Koszmider, *Some applications of set theory in Banach spaces*

  15:30–16:00 Alberto Salguero-Alarcón, *Almost disjoint family ties in Banach spaces*

  coffee break

  16:30–17:00 Witold Marciszewski, κ*-Corson compacta and function spaces*

  17:00–18:00 Mikołaj Krupski, κ*-pseudocompactness and uniform homeomorphisms of function spaces*

  18:00–18:30 Piotr Szewczak, *Perfectly meager sets and the Hurewicz property*

- Wednesday – September 6th

  12:00–12:30 Eliza Jabłońska, *Applications of null-finite sets in set-valued map*

  12:30–13:00 Joanna Garbulińska-Węgrzyn, *Erdös-like spaces as Fraïssé limits in some metric categories*

  13:00–13:30 Taras Banakh, *Banakh spaces and their Geometry*

- Thursday – September 7th

  14:00–15:00 Grigor Sargsyan, *Forcing over models of determinacy*

  15:00–15:30 Antonio Avilés, *Examples of compact* $L\Sigma(\leqslant \omega)$*-spaces*

  15:30–16:00 Maciej Korpalski, *Free dimension and isomorphisms of spaces of continuous functions*

  coffee break

  16:30–17:00 Jacek Tryba, *Different kinds of density ideals*

  17:00–17:30 Łukasz Mazurkiewicz, *Ideal analytic sets*

  17:30–18:00 Robert Rałowski, *The Baire theorem, an analogue of the Banach fixed point theorem*

- Friday – September 8th

  14:00–14:30 Szymon Żeberski, *Eggleston meets Mycielski, measure case*

  14:30–15:00 Marcin Michalski, *Eggleston and Mycielski-like theorems for category*

  15:00–15:30 Damian Głodkowski, *A Banach space* $C(K)$ *reading the dimension of* K

  15:30–16:00 Kamil Ryduchowski, *Antiramsey colorings and geometry of Banach spaces*

Forcing axioms beyond $H(\omega_2)$

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Abstract

Forcing axioms assert the existence, given a suitable class Γ of forcing notions, of "sufficiently generic" filters on $\mathcal{P}$, for every $\mathcal{P} \in \Gamma$. In the classical setting, "sufficiently generic" means meeting all members of $\mathcal{D}$, for any family $\mathcal{D}$ of $\aleph_1$-dense subsets of $\mathcal{P}$ fixed in advance. The family of classical forcing axioms, ordered by implication, is known to have a top element, which successfully decides the theory of $H(\omega_2)$. I will survey old and new results concerning the general problem of extending this picture, or finding analogues of it, beyond $H(\omega_3)$. In particular, I will discuss the prospects of finding strong forcing axioms at the level of $H(\omega_3)$ or higher up.

Examples of compact $L\Sigma(\leqslant \omega)$-spaces

Antonio Avilés

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joint work with Mikołaj Krupski

Abstract

The class of $L\Sigma(\leqslant \omega)$-spaces was introduced in 2006 by Kubiś, Okunev and Szeptycki as a natural refinement of the classical and important notion of Lindelöf Σ-spaces. Compact $L\Sigma(\leqslant \omega)$-spaces were considered earlier, under different names, in the works of Tkachuk and Tkachenko in relation to metrizably fibered compacta. In this paper we give counterexamples to several open questions about compact $L\Sigma(\leqslant \omega)$-spaces that are scattered in the literature. Among other things, we refute a conjecture of Kubiś, Okunev and Szeptycki by constructing a separable Rosenthal compactum which is not an $L\Sigma(\leqslant \omega)$-space. We also give insight to the structure of first-countable $(K)L\Sigma(\leqslant \omega)$-compacta.

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Banakh spaces and their Geometry

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Abstract

In one question posed at `Mathoverflow` I asked whether the real line is a unique metric space (X, d) such that for every positive real number r, every r-sphere $S(c; r) := \{x \in X : d(x, c) = r\}$ in X has cardinality 2 and diameter $2r$. Will Brian proved that this is indeed true in the realm of complete metric spaces. On the other hand, Pietro Mayer suggested an idea of constructing an example showing that without completeness such a characterization of the real line is not true. This motivated studying such metric spaces in more details. To make as much confusion as possible, Will Brian suggested to call a metric space (X, d) *Banakh* if for every point $c \in X$ and real number $r \in d[X^2]$ the sphere $S_d(c; r)$ has cardinality 2 and diameter $2r$. In the talk we shall discuss Banakh spaces and their amusing geometry. In particular, we prove that for every points x, y of a Banakh space X there exists a unique subspace Z of X that contains the points x, y and is isometric to the discrete line $\mathbb{Z}\cdot d(x, y)$. Using this geometric fact, we prove that a metric space (X, d) is isometric to a subgroup G of the additive group $\mathbb{Q}$ of rational numbers if and only if X is a Banakh space with $d[X^2] = G_+ := \{x \in G : x \geqslant 0\}$. Generalizing the mentioned result of Will Brian, we prove that a metric space X is isometric to the real line if and only if X is a complete Banakh space such that $G_+ \subseteq d[X^2]$ for some non-cyclic subgroup $G \subseteq \mathbb{Q}$. We prove that every Banakh space (X, d) with $d[X^2] \subseteq \mathbb{Q}$ is isometric to some subgroup of $\mathbb{Q}$. Every Banakh space X satisfies the "rational" axiom of segment construction: for every $x, y \in X$ and every $r \in d[X^2] \cap \mathbb{Q}\cdot d(x, y)$ there exists a unique point $z \in X$ such that $d(y, z) = r$ and $d(x, z) = d(x, y) + d(y, z)$. This result implies that for every Banakh space X, the set $d[X^2] \cap \mathbb{Q}$ is a submonoid of the group $\mathbb{Q}$. Yet, for every closed discrete submonoid M of $\mathbb{Q}$ there exists a countable Banakh space (X, d) such that $d[X^2] \cap \mathbb{Q} = M$. Also for every nonzero cardinal $\kappa \leqslant \mathfrak{c}$, the Hilbert space $\ell_2(\kappa)$ contains a discrete subgroup H which is a complete Banakh space of cardinality $|H| = \max\{\kappa, \omega\}$, and a dense $\mathbb{Q}$-linear subspace L such that L is a Banakh space with $d[L^2] = \mathbb{R}_+$.

[1] Banakh T., *Banakh spaces and their Geometry*, arXiv:2305.07354 (2023).

Erdös-like spaces as Fraïssé limits in some metric categories

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Abstract

Considering a category of finite metric spaces with morphisms defined as pairs consisting of isometries and contractions, we get some universal structures as a Fraïssé limit. We investigate connections between obtained objects and Erdös space (or its countable infinite power). Moreover, we prove that Erdös space has Boolean group structures.

A Banach space $C(K)$ reading the dimension of K

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Abstract

We show that (assuming Jensen's diamond principle $\diamondsuit$) for every natural number n there is a Banach space $C(K)$ of continuous functions on a compact Hausdorff space K, such that for every L, if $C(L)$ is isomorphic to $C(K)$ then $\dim L = n$. Constructed spaces are examples of Banach spaces of continuous functions with few operators, which were introduced by Koszmider in [2]. The talk will be based on [1].

[1] Głodkowski D., *A Banach space C(K) reading the dimension of K*, Journal of Functional Analysis 285 (2023), no. 4.

[2] Koszmider P., *Banach spaces of continuous functions with few operators*, Mathematische Annalen 330 (2004), no. 1, 151–183.

Applications of null-finite sets in set-valued map

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Abstract

In [1] the following notion of a "small" set has been introduced: a subset A of a topological group X is called *null-finite* if there exists a sequence $(x_n)_{n\in\omega}$ convergent to 0 in X such that for every $x \in X$ the set $\{n \in \omega : x + x_n \in A\}$ is finite. It turns out that such type of "small" sets can be applied in the theory of set-valued maps.

We present that in some special classes of set-valued maps, i.e.:

- of subadditive set-valued maps,
- of weakly subadditive set-valued maps,
- of midconvex set-valued maps,

upper boundedness on a non-null-finite set implies some kind of regularity of a set-valued map like continuity or local boundedness at every point.

[1] Banakh T., Jabłońska E., *Null-finite sets in topological groups and their applications*, Israel Journal of Mathematics 230 (2019), 361–386.

Free dimension and isomorphisms of spaces of continuous functions

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Abstract

In [1], Marděsić stated a conjecture that whenever a product of d linearly ordered compact spaces can be mapped onto a product of $d + s$ separable infinite compact spaces $K_1, \dots, K_{d+s}$ with $s \geqslant 1$, then there are at least $s + 1$ metrizable factors K_j . This conjecture was turned into a theorem in [1] using a new notion of free dimension. Such a notion can be used to approach other problems, one of them is a question of isomorphisms between Banach spaces. Surjections between compact spaces are naturally lifting to embeddings of Banach spaces of continuous functions, so free dimension can possibly be applied to spaces of functions defined on e.g. products of compact lines. For products of separable compact lines this question was already solved in [2], but the nonseparable case is an open problem.

[1] Marděsić S., *Mapping products of ordered compacta onto products of more factors*, Glasnik Matematicki Series III 5 (1970), no. 25, 163–170.

[2] Martínez-Cervantes G., Plebanek G., *The Marděsić Conjecture and free products of Boolean algebras.*, Proceedings of the American Mathematical Society 147 (2019), 1763–1772.

[3] Michalak A., *On Banach spaces of continuous functions on finite products of separable compact lines*, Studia Mathematica 251 (2020), 247–275.

Some applications of set theory in Banach spaces

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Abstract

We will present selected recent applications of set-theoretic methods in Banach spaces.

κ-pseudocompactness and uniform homeomorphisms of function spaces

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Abstract

A Tychonoff space X is called *κ-pseudocompact* if for every continuous mapping f of X into $\mathbb{R}^\kappa$ the image $f(X)$ is compact. This notion generalizes pseudocompactness and gives a stratification of spaces lying between pseudocompact and compact spaces. It is well known that pseudocompactness of X is determined by the uniform structure of the function space $C_p(X)$ of continuous real-valued functions on X endowed with the pointwise topology. Answering a question of Arhangel'skii, we show that analogous assertion is true for κ-pseudocompactness. Our proof relies on fact that κ-pseudocompact can be conveniently characterized by the way X is positioned in its Čech-Stone compactification βX. We shall mention other results concerning the linear-topological structure of the space $C_p(X)$, where this idea can also be applied. The talk is based on the recent paper [1].

[1] Krupski M., *On κ-pseudocompactess and uniform homeomorphisms of function spaces*, Results in Mathematics 78 (2023), no. 154.

On Pelczynski universal space: double Fraïssé classes

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joint work with Jamal K. Kawac
and joint work in progress with S. Todorcevic

Abstract

We present an isomorphic version of the well-known Pelczynski universal space as a Fraïssé limit of what we call a double Fraïssé class. We will discuss the Fraïssé and the KPT correspondence, and the general theory involving other types of structures.

κ-Corson compacta and function spaces

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joint work with Grzegorz Plebanek and Krzysztof Zakrzewski

Abstract

Let κ be an infinite cardinal number. A compact space K is κ-Corson compact if, for some set Γ, K is homeomorphic to a subset of the Σ_κ-product of real lines

$$\Sigma_\kappa(\mathbb{R}^\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma : x(\gamma) \neq 0\}| < \kappa\}.$$

Obviously, the well known class of Corson compact spaces coincides with the class of ω_1-Corson compact spaces.

We will present some recent results concerning κ-Corson compact spaces and related function spaces.

[1] Marciszewski W., Plebanek G., Zakrzewski K., *Digging into the classes of κ-Corson compact spaces*, arXiv:2107.02513v4.

Ultraproducts and Michael spaces

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Abstract

A Lindelöf space M is a *Michael space* if $M \times \mathbb{N}^{\mathbb{N}}$ is not Lindelöf. The first example of such spaces was constructed by E. Michael [1] using the Continuum Hypothesis and since then, many other consistent examples were constructed. One of them was by J. Moore [2] in which he gave a general framework to construct one of this spaces; the author used this framework to construct a Michael space under $\mathfrak{d} = \mathrm{cov}(\mathcal{M})$. In this talk, we will talk about this framework and we will use the structure of the ultraproduct given by a selective ultrafilter (plus $\varepsilon \geqslant 0$) to construct a Michael space. In particular, we will prove that there is a Michael space after forcing with $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$.

[1] Michael E., *The Product Of A Normal Space And A Metric Space Need Not Be Normal*, Bulletin of the American Mathematical Society 69 (1963), 375–376.

[2] Moore J.T., *Some of the Combinatorics Related to Michael's Problem*, Proceedings of the American Mathematical Society 127 (1999), no. 8, 2459–2467.

Ideal analytic sets

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joint work with Szymon Żeberski

Abstract

For $B \subseteq \omega$ we write

$$FS(B) = \left\{ \sum_{n \in F} n : F \subseteq B \text{ is nonempty and finite} \right\}.$$

A set $A \subseteq \omega$ is called an *IP-set* if there is an infinite $B \subseteq A$ satisfying $FS(B) \subseteq A$. A family of non-*IP-sets* is called *Hindmann ideal*.

Showing analytic completeness of given analytic set A is one of few options to prove, that A is not Borel. In the talk we will discuss results concerning examples of ideals on ω (treated as a subset of Cantor space) constructed in a similar way to Hindmann ideal.

[1] Filipów R., *On Hindman spaces and the Bolzano-Weierstrass property*, Topology and its Applications 160 (2013), no. 15, 2003–2011.

Eggleston and Mycielski-like theorems for category

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joint work with Robert Rałowski and Szymon Żeberski ([2] and [3])

Abstract

Let us recall the following the two following theorems on inscribing special kind rectangles and squares into large subsets of the plane.

Eggleston Theorem [1]
For every conull set $F \subseteq [0,1]^2$ there are a perfect set $P \subseteq [0,1]$ and conull $B \subseteq [0,1]$ such that $P \times B \subseteq F$.

Mycielski Theorem [4]
For every comeager or conull set $X \subseteq [0,1]^2$ there exists a perfect set $P \subseteq [0,1]$ such that $P \times P \subseteq X \cup \Delta$, where $\Delta = \{(x,x) : x \in [0,1]\}$.

We will consider the category variant of the former (comeager instead of conull) in the Cantor space 2^ω and its strengthening via replacing a perfect set with a body of some type of a perfect tree. Mainly we will focus on uniformly perfect trees, Silver trees and Spinas trees. Moreover we will explore the possibility of conjoining the above theorems by demanding that for a comeager set $G \subseteq 2^\omega \times 2^\omega$ there is a comeager set $B \subseteq 2^\omega$ and a tree T of certain kind such that $[T] \times B \subseteq G$ (modulo diagonal) and $[T] \subseteq B$.

[1] Eggleston H. G., *Two measure properties of Cartesian product sets*, The Quarterly Journal of Mathematics 5 (1954), 108–115.

[2] Michalski M., Rałowski R., Żeberski Sz., *Around Eggleston theorem*, arXiv:2307.07020 (2023).

[3] Michalski M., Rałowski R., Żeberski Sz., *Mycielski among trees*, Mathematical Logic Quarterly 67 (2021), 271–281.

[4] Mycielski J., *Algebraic independence and measure*, Fundamenta Mathematicae 61 (1967), 165–169.

The Baire theorem, an analogue of the Banach fixed point theorem

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joint work with M. Morayne

Abstract

We prove that if X is a T_1 second countable compact space, then X is a Baire space if and only if every open subset of X contains a closed subset with nonempty interior. We also prove an analogue of Banach's fixed point theorem for all T_1 compact spaces. Applying the analogue of Banach's fixed point theorem we prove the existence of unique attractors for so called contractive iterated function systems whose Hutchinson operators are closed in compact T_1 spaces.

[1] Morayne M., Rałowski R., *The Baire Theorem, an Analogue of the Banach Fixed Point Theorem and Attractors in Compact Spaces*, Bulletin des Sciences Mathematiques 183 (2023).

Antiramsey colorings and geometry of Banach spaces

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joint work with Piotr Koszmider

Abstract

With every coloring $c\colon [\omega_1]^2 \to \{0,1\}$ we shall associate some nonseparable Banach space X_c. The talk will focus on the following problem: how to translate various combinatorial properties of the coloring c into geometric properties of the space X_c. We will show, among others, that while MA + ¬ CH implies that the geometry of the spaces X_c is quite regular, some interesting phenomena occur when c has some strong antiramsey properties (so strong that the existence of such c is independent of ZFC).

[1] Guzmán O., Hrušák M., Koszmider P., *Almost disjoint families and the geometry of nonseparable spheres*, arXiv:2212.05520 (2023).

[2] Hájek P., Kania T., Russo T., *Separated sets and Auerbach systems in Banach spaces*, Transactions of the American Mathematical Society 373 (2020), 6961–6998.

[3] Koszmider P., Ryduchowski K., *Equilateral and separated sets in some Hilbert generated Banach spaces*, arXiv:2301.07413 (2023).

[4] Koszmider P., Wark H.M., *Large Banach spaces with no infinite equilateral sets*, Bulletin of the London Mathematical Society 54 (2022), 2066–2077.

Almost disjoint family ties in Banach spaces

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Abstract

An almost disjoint family of subsets of $\mathbb{N}$ is just a family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ such that every two different members of $\mathcal{A}$ have finite intersection. Almost disjoint families produce nice examples of Banach spaces of continuous real-valued functions on compact spaces, or $C(K)$-spaces, for short. Indeed, every almost disjoint family $\mathcal{A}$ produces a compact space $K_{\mathcal{A}}$ –the *Alexandrov-Urysohn* space associated to $\mathcal{A}$– and so we obtain the Banach space $C(K_{\mathcal{A}})$ of continuous functions on $K_{\mathcal{A}}$. However, if we are willing to exploit the machinery of infinite combinatorics to carefully produce peculiar almost disjoint families, then we open door to exotic Banach spaces that serve as counterexamples for both natural and important questions in the theory of $C(K)$-spaces.

We cannot resist to mention one such question: the so-called *complemented subspace problem*, which asked whether every complemented subspace of a $C(K)$-space must be, again, a $C(K)$-space. After more than 50 years, a counterexample was produced in [1] with the invaluable help of almost disjoint families. In doing so, other mysteries about subspaces and quotients of $C(K)$-spaces were also explained. This talk is dedicated to explore several such mysteries, with special attention to the ideas leading to the construction of the counterexample for the complemented subspace problem.

[1] Plebanek G., Salguero-Alarcón S., *The complemented subspace problem for $C(K)$-spaces: A counterexample*, Advances in Mathematics 426 (2023).

Forcing over models of determinacy

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Abstract

We will survey some recent results obtained by forcing over models of determinacy. In one such model, the restriction of the omega-club filter on the first three uncountable cardinals is an ultrafilter in HOD (this answers a question of Ben Neria and Hayut, and is a joint work with Takehiko Gappo), and in another square fails at omega_3 (this is a joint work with Larson). The first result is related to Woodin's HOD Conjecture and the second is related to the iterability problem.

Perfectly meager sets and the Hurewicz property

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joint work with Tomasz Weiss and Lyubomyr Zdomskyy

Abstract

We work in the Cantor space with the usual group operation $+$. A set X is *perfectly meager in the transitive sense* if for any perfect set P there is an F_σ-set F containing X such that for every point t the intersection $F \cap (t+P)$ is meager in the relative topology of $t+P$. A set X is *Hurewicz* if for any sequence $\mathcal{U}_0, \mathcal{U}_1, \ldots$ of open covers of X, there are finite families $\mathcal{F}_0 \subseteq \mathcal{U}_0, \mathcal{F}_1 \subseteq \mathcal{U}_1, \ldots$ such that the family $\{\bigcup \mathcal{F}_n : n \in \omega\}$ is a γ-cover of X, i.e., the sets $\{n : x \notin \bigcup \mathcal{F}_n\}$ are finite for all points $x \in X$. Nowik proved that each Hurewicz set which cannot be mapped continuously onto the Cantor set is perfectly meager in the transitive sense. We present results related to the question, whether the same assertion holds for each Hurewicz set with no homeomorphic copy of the Cantor set inside.

The research was funded by the National Science Center, Poland and the Austrian Science Found under the Weave-UNISONO call in the Weave programme, project:
Set-theoretic aspects of topological selections 2021/03/Y/ST1/00122.

Different kinds of density ideals

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Abstract

We consider several classes of ideals described by some densities. We present connections between Erdős-Ulam, density, matrix summability and generalized density ideals, compare these classes of ideals and show that a certain inaccuracy in Farah's definition of density ideals leads to Farah's characterization when density ideals are Erdős-Ulam ideals being incorrect.

We also show that one of these classes of ideals can be used to characterize ideals given by nonpathological submeasures as well as provide solution to the Problem 5 from The Scottish Book.

Eggleston meets Mycielski, measure case

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joint work with M. Michalski and R. Rałowski

Abstract

The motivation of this work are the two classical theorems on inscribing rectangles and squares into large subsets of the plane, namely Eggleston Theorem and Mycielski Theorem.

We proved that every conull subset of the plane contains a rectangle $[T] \times H$, where T is a Spinas tree containing a Silver tree and H is conull. Moreover we obtained a common generalization of Eggleston Theorem and Mycielski Theorem stating that every conull subset of the plane contains a rectangle $[T] \times H$ modulo diagonal, where T is a uniformly perfect tree, H is conull and $[T] \subseteq H$.

[1] Michalski M., Rałowski R., Żeberski S., *Around Eggleston theorem*, arXiv: 2307.07020 (2023).