Thematic section

AGNT

Arithmetic Geometry and Number Theory

ORGANIZERS:

Grzegorz Banaszak (Uniwersytet Adama Mickiewicza, Poznań)

Francesc Bars Cortina (Universitat Autonoma de Barcelona)

Wojciech Gajda (Uniwersytet Adama Mickiewicza, Poznań)

SCHEDULE OF THE SECTION
Arithmetic Geometry and Number Theory

- Monday – September 4th

  16:00–16:05 WELCOME

  16:05–17:00 Enric Nart, *Defect in extensions of valuations*

  coffee break

  17:30–18:30 Piotr Achinger, *Maps of affine varieties and wild ramification*

  18:30–19:00 Jędrzej Garnek, *Cohomologies of p-group covers*

- Tuesday – September 5th

  14:30–15:30 Masha Vlasenko, *Frobenius structure and p-adic zeta function*

  15:30–16:00 Álvaro Serrano Holgado, *The generalized Zeta functions of a linear recurrence sequence*

  coffee break

  16:30–17:30 Luis M. Navas, *On a relation between power and Dirichlet series*

  17:30–18:00 Jolanta Marzec-Ballesteros, *Bounds on Fourier coefficients and global sup-norms for Siegel cusp forms of degree* 2

  18:00–18:30 Tomasz Jędrzejak, *Ranks of quadratic twists of Jacobians of generalized Mordell curves*

- Wednesday – September 6th

  12:00–13:00 Zbigniew Hajto, *Real and p-adic Picard-Vessiot extensions*

  13:00–13:30 Teresa Crespo, *Hopf Galois structures*

- Thursday – September 7th

  14:00–15:00 Daniel Macias Castillo, *The refined class number formula for Drinfeld modules*

  15:00–15:30 Bartosz Naskręcki, *Higher moments of families of elliptic curves*

  15:30–16:00 Antonio Rojas-León, *Independence of Gauss sums*

  coffee break

  16:30–17:15 Xavier Guitart, *A quaternionic construction of p-adic singular moduli*

  17:15–18:00 Santiago Molina Blanco, *Waldspurger formulas in higher cohomology*

  18:00 CLOSING

- Friday – September 8th

  14:00–15:00 Joan Carles Lario, *When the Birch–Swinnerton–Dyer conjecture was not remunerated*

Maps of affine varieties and wild ramification

Piotr Achinger

IMPAN
Warszawa
email: pachinger@impan.pl

joint work with Jakob Stix

Abstract

We show that if two nonconstant maps between two affine varieties defined over two possibly different perfect fields induce the same maps on the etale fundamental group, then they are in fact the same up to a power of Frobenius. In fact, it is enough to consider etale cohomology with $\mathbb{F}_p$-coefficients.

When the Birch–Swinnerton–Dyer conjecture was not remunerated

Joan Carles Lario

UPC
email: joan.carles.lario@upc.edu

Abstract

In this talk I will present a personal overview on the chain of events that led to the conjecture of Birch and Swinnerton-Dyer.

Special emphasis will be placed on the role played by Kurt Heegner, through his famous article where Gauss's problem on the determination of imaginary quadratic fields with number of classes one is solved, along with relevant results on the classical problem of congruent numbers.

We will finish with a note on the determination of quadrilateral numbers introduced by Kummer that was proposed to Heegner by his teacher Schwatrz (the son-in-law of Kummer).

Hopf Galois structures

Teresa Crespo

Universitat de Barcelona
Departament de Matemàtiques i Informàtica
email: teresa.crespo@ub.edu

Abstract

For a Galois extension L/K with Galois group G, the action of G on L by K-automorphisms extends to an action of the group algebra $K[G]$ on L, by endomorphisms of L as a K-vector space. A Hopf Galois structure on a field extension L/K consists in a K-Hopf algebra H together with an action of H on L, satisfying conditions which mimic the properties of the action of $K[G]$ in the Galois case. For a finite separable extension L/K, Greither and Pareigis [3] obtained that Hopf Galois structures may be determined in terms of group theory. In my talk I will introduce Hopf Galois structures on finite separable field extensions and present some recent results on extensions of prime power degree ([1], [1]).

[1] Crespo T., *Automatic realization of Hopf Galois structures*, Journal of Algebra and Its Applications 21 (2022), no. 2.

[2] Crespo T., Salguero M., *Hopf Galois structures on separable field extensions of odd prime power degree*, Journal of Algebra 519 (2019), 424–439.

[3] Greither C., Pareigis B., *Hopf Galois theory for separable field extensions*, Journal of Algebra 106 (1987), 239–258.

Cohomologies of p-group covers

Jędrzej Garnek

Institute of Mathematics of Polish Academy of Sciences, Warsaw
email: jgarnek@amu.edu.pl

Abstract

Studying cohomology of a variety with an action of a finite group is a classical and well-researched topic. However, most of the previous results focus either on the tame ramification case, on some special groups, or on specific curves. In the talk, we will consider the case of a curve over a field of characteristic p with an action of a finite p-group. Our research suggests that the Hodge and de Rham cohomologies decompose as sums of certain 'local' and 'global' parts. The global part should be determined by the 'topology' of the cover, while the local parts should depend only on an analytical neighborhood of the fixed points of the action. In fact, the local parts should come from cohomologies of Harbater–Katz–Gabber curves, i.e. covers of the projective line ramified only over ∞. During the talk, we present our results related to this conjecture. As an application, we compute the de Rham cohomologies of $\mathbb{Z}/p$-covers and Klein four covers.

A quaternionic construction of p-adic singular moduli

Xavier Guitart

Universitat de Barcelona
email: xevi.guitart@gmail.com

Abstract

Rigid meromorphic cocycles were introduced by Henri Darmon and Jan Vonk as a conjectural p-adic extension of singular moduli over real quadratic base fields. They are certain cohomology classes of SL_2 that can be evaluated at real quadratic quantities and the resulting values are conjectured to be algebraic. In this talk I will explain joint work with Marc Masdeu and Xavier Xarles in which we propose a similar construction of cohomology classes in quaternion algebras over totally real fields F. These classes can be evaluated at elements of quadratic extensions K/F, and we conjecture that the resulting values belong to abelian extensions of K. This conjecture is supported by numerical evidence.

Real and p-adic Picard-Vessiot extensions

Zbigniew Hajto

Jagiellonian University
Faculty of Mathematics and Computer Science
email: zbigniew.hajto@uj.edu.pl

Abstract

I will survey Galois theory for partial differential systems defined over formally real differential fields with a real closed field of constants and over formally p-adic differential fields with a p-adically closed field of constants. For an integrable partial differential system defined over such a field, there exists a formally real (resp. formally p-adic) Picard-Vessiot extension. I will comment on the uniqueness result for these Picard-Vessiot extensions and the Galois correspondence theorem in this setting. I will explain the application of this theorem to characterise formally real Liouvillian extensions of real partial differential fields with a real closed field of constants by means of split solvable linear algebraic groups. In this context, I will discuss the topological properties of real Liouville functions and relate them with the concept of tame topology in the sense of Grothendieck and Khovanskii. Finally, I will discuss some possibilities for further development of this theory.

[1] Crespo T., Hajto Z., Mohseni R., *Real Liouvillian Extensions of Partial Differential Fields*, SIGMA 17 (2021), no. 95.

[2] Crespo T., Hajto Z., van der Put M., *Real and p-adic Picard-Vessiot fields*, Mathematische Annalen 365 (2016), 93–103.

Ranks of quadratic twists of Jacobians of generalized Mordell curves

Tomasz Jędrzejak

University of Szczecin
Institute of Mathematics
email: tjedrzejak@gmail.com

Abstract

Consider a two-parameter family of hyperelliptic curves $C_{q,b} : y^2 = x^q - b^q$ defined over $\mathbb{Q}$, and their Jacobians $J_{q,b}$ where q is an odd prime and without loss of generality b is a non-zero squarefree integer. The curve $C_{q,b}$ is a quadratic twist by b of $C_{q,1}$ (a generalized Mordell curve of degree q). First, we obtain a few upper bounds for the ranks e.g., if $q \equiv 1 \pmod 4$ and any prime divisor of $2b$ not equal to q is a primitive root modulo q then $\operatorname{rank} J_{q,b}(\mathbb{Q}) \leqslant (q-1)/2$. Then we focus on $q = 5$ and get the best possible bound (by 1) or even the exact value of rank (0). In particular, we found infinitely many b with any number of prime factors such that $\operatorname{rank} J_{5,b}(\mathbb{Q}) = 0$. We deduce as conclusions the complete list (or the bounds for the number) of rational points on $C_{5,b}$ in such cases. Finally, we found for any given q infinitely many non-isomorphic curves $C_{q,b}$ such that $\operatorname{rank} J_{q,b}(\mathbb{Q}) \geqslant 1$.

[1] Jędrzejak T., *Ranks of quadratic twists of Jacobians of generalized Mordell curves*, under review.

[2] Juyal A., Moody D., Roy B., *On ranks of quadratic twists of a Mordell curve*, The Ramanujan Journal 59 (2022), 31–50.

The refined class number formula for Drinfeld modules

Daniel Macias Castillo

Universidad Autónoma de Madrid
email: daniel.macias@icmat.es

joint work with María Inés de Frutos Fernández
and Daniel Martínez Marqués

Abstract

In 2012, Taelman proved an analogue of the Analytic Class Number Formula, for the Goss L-functions that are associated to Drinfeld modules. He also explicitly stated that 'it should be possible to formulate and prove an equivariant version' of this formula.

We formulate and prove an equivariant, or 'refined', version of Taelman's formula.

As a concrete consequence of our general approach, we also derive explicit consequences for the Galois structure of Taelman class groups.

Bounds on Fourier coefficients and global sup-norms for Siegel cusp forms of degree 2

Jolanta Marzec-Ballesteros

Kazimierz Wielki University, Bydgoszcz
UAM, Poznań
email: jmarzec@ukw.edu.pl

Abstract

Siegel modular forms of degree n are a natural generalisation of classical modular forms (of degree 1). They are holomorphic functions invariant under the action of (subgroups of) $\mathrm{Sp}_{2n}(\mathbb{Z})$, possess Fourier expansion and - when cuspidal - are square-integrable. One of the most basic and yet unsolved problems concerns the growth of their Fourier coefficients. It is known as Ramanujan-Petersson conjecture when $n = 1$ and as Resnikoff-Saldaña conjecture when $n \geqslant 2$. Deligne's proof of the first conjecture had a significant impact on many problems in mathematics, cf. [3], including an optimal solution to a sup-norm problem given by Xia in [4]. When $n > 1$ hardly anything is known.

In the talk we focus on the case $n = 2$. We will present a special case of Gan-Gross-Prasad conjecture, proven by Furusawa and Morimoto [2], and explain how it led us to the best current bounds on Fourier coefficients, both unconditionally and under GRH. As a consequence we will derive a global sup-norm bound for Siegel cusp forms of degree 2 invariant by $\mathrm{Sp}_4(\mathbb{Z})$ in terms of their weight and L^2-norm.

[1] Comtat F., Marzec-Ballesteros J., Saha A., *Bounds on Fourier coefficients and global sup-norms for Siegel cusp forms of degree 2*, arXiv:2307.07376 (2023).

[2] Furusawa M., Morimoto K., *On the Gross-Prasad conjecture with its refinement for* $(\mathrm{SO}(5), \mathrm{SO}(2))$ *and the generalized Böcherer conjecture*, arXiv:2205.09503 (2022).

[3] Sarnak P., *Some applications of modular forms*, Cambridge University Press, Cambridge (1990).

[4] Xia H., *On* L^∞ *norms of holomorphic cusp forms*, Journal of Number Theory 124 (207), p. 325–327.

Waldspurger formulas in higher cohomology

Santiago Molina Blanco

Universitat de Lleida
email: santiago.molina@udl.cat

Abstract

The classical Waldspurger formula, which computes periods of quaternionic automorphic forms in maximal torus, has been used in a wide variety of arithmetic applications, such as the Birch and Swinnerton-Dyer conjecture in rank 0 situations. This is why this formula is considered the rank 0 analogue of the celebrated Gross-Zagier formula.

On the other hand, Eichler-Shimura correspondence allows us to interpret this quaternionic automorphic form as a cocycle in higher cohomology spaces of certain arithmetic groups. In this way we can realize the corresponding automorphic representation in the etale cohomology of certain Shimura varieties. In this work we find a formula, analogous to that of Waldspurger, which relates cap-products of this cocycle and fundamental classes associated with maximal torus with special values of Rankin-Selberg L-functions.

Defect in extensions of valuations

Enric Nart

Universitat Autònoma de Barcelona
email: enric.nart@uab.cat

joint work with Josnei A. Novacoski

Abstract

Given a valued field (K, v) and a simple finite extension L/K, there are a finite number of valuations $w_1, \dots, w_r$ on L extending v. To each valuation w_i we may associate natural numbers $e(w_i/v)$, $f(w_i/v)$, $d(w_i/v)$ called the *ramification index*, *inertia degree* and *defect*, respectively. These numbers are linked by the formula:

$$[L\colon K] = \sum_{i=1}^{r} e(w_i/v) f(w_i/v) d(w_i/v).$$

The choice of a generator of L/K determines an onto ring homomorphism $K[x] \twoheadrightarrow L$ leading to a reinterpretation of each valuation w on L as a valuation on $K[x]$:

$$\nu\colon K[x] \longrightarrow\!\!\!\!\rightarrow L \xrightarrow{w} \Gamma \cup \{\infty\},$$

where Γ is the value group of w. The Mac Lane-Vaquié theory expresses ν as the last step of a finite chain of *augmentations* of valuations on $K[x]$:

$$\mu_0 \longrightarrow \mu_1 \longrightarrow \cdots \longrightarrow \mu_{\ell-1} \longrightarrow \mu_\ell = \nu,$$

starting with a very simple (degree-one) valuation μ_0.

We shall define a natural number $d(\mu_n \to \mu_{n+1})$ associated to each augmentation, so that the defect of the extension w/v can be expressed as:

$$d(w/v) = d(\mu_0 \to \mu_1) \cdots d(\mu_{\ell-1} \to \mu_\ell).$$

This result was proved by Vaquié under the assumption that (K, v) is henselian.

Higher moments of families of elliptic curves

Bartosz Naskręcki

Adam Mickiewicz University, Poznań
Faculty of Mathematics and Computer Science
email: bartosz.naskrecki@amu.edu.pl

joint work with Matija Kazalicki ([2], [1])

Abstract

In this talk we will discuss some new developments in higher moment sums of 1-parametric families of elliptic curves. These sums have connections to modular forms and algebraic curves. I will sketch a result about the second moment of cubic curves leading to a connection with intermediate Jacobians in threefolds. Next we will discuss proofs of modularity of certain rigid Calabi-Yau threefolds which uses directly higher moments, universal families of elliptic curves and Deligne's results, avoiding completely the standard approach via Faltings-Serre method.

[1] Kazalicki M., Naskręcki B., *Second moments and the bias conjecture for the family of cubic pencils*, arXiv:2012.11306 (2021).

[2] Kazalicki M., Naskręcki B., *Diophantine triples and K3 surfaces*, Journal of Number Theory 236 (2022), 41–70.

On a relation between power and Dirichlet series

Luis M. Navas

Universidad de Salamanca
Dpto. de Matemáticas
email: navas@usal.es

Abstract

Given any convergent Laurent series with finite singular part, $f(z) = \sum_{n=-k}^{\infty} a_n z^n$ ($k \geqslant 0$), we can construct via a truncated Mellin transform a meromorphic function $F(s)$ on $\mathbb{C}$ whose poles are simple and contained in the finite set $\{1, 2, \ldots, k\}$, with residues $\operatorname{Res}(F; n) = a_{-n}/(n-1)!$ and values $F(-n) = (-1)^n n!\, a_n$ for integer $n \geqslant 0$. Such an F is far from unique, though.

When $f(z) = H(e^{-z})e^{-z}$ with $H(z)$ analytic on the complex open unit disk $\mathbb{D}$, there is a corresponding $F(s)$ which is a complete Mellin transform and constitutes a "natural" choice. If the Taylor series at $z = 0$ of $H(z)$ is $\sum_{n=0}^{\infty} h_n z^n$, the existence of a meromorphic continuation of $H(z)$ to $z = 1$ is reflected in the existence of the meromorphic continuation of the Dirichlet series $D(s) = \sum_{n=1}^{\infty} h_{n-1} n^{-s}$ to $\mathbb{C}$, given by $F(s)$, and having several special properties. We provide many examples, including one which is reminiscent of [6].

This construction also allows us to conclude that certain power series on the unit disk do *not* have a meromorphic continuation to $z = 1$, by checking that the corresponding Dirichlet series does not satisfy one or more of the properties the theorem predicts it should have. This kind of result is weaker but also easier (using purely complex analytic techniques) than those of [2] or [3].

The results and methods presented here are contained in the papers [4, 5]. The overall philosophy is related to the non-rigorous formula known as "Ramanujan's Master Theorem" [1].

[1] Amdeberhan T., Espinosa O., Gonzalez I., Harrison M., Mall V. H., Straub A., *Ramanujan's master theorem*, Ramanujan Journal 29 (2012), 103–120.

[2] Bell J.P., Bruin N., Coons M., *Transcendence of generating functions whose coefficients are multiplicative*, Transactions of the American Mathematical Society 364 (2012), no. 2, 933–959.

[3] Knill O., Lesieutre J., *Analytic continuation of Dirichlet series with almost periodic coefficients*, Complex Analysis and Operator Theory 6 (2012), no. 1, 237–255.

[4] Navas L.M., Ruiz F.J., Varona J.L., *Appell polynomials as values of special functions*, Journal of Mathematical Analysis and Applications 459 (2018), no. 1, 419–436.

[5] Navas L.M., Ruiz F.J., Varona J.L., *A connection between power series and Dirichlet series*, Journal of Mathematical Analysis and Applications 493 (2021).

[6] Serrano H.Á., Navas L.M., *The zeta function of a recurrence sequence of arbitrary degree*, Mediterranean Journal of Mathematics 20 (2023), no. 224.

Independence of Gauss sums

Antonio Rojas-León

Universidad de Sevilla
email: arojas@us.es

Abstract

Given a finite field $k/\mathbb{F}_p$ and n r-tuples $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{Z}^r$ we consider the question of the simultaneous distribution of the normalized Gauss sums $G(\chi^{\mathbf{a}_1}), \ldots, G(\chi^{\mathbf{a}_n})$ in $(S^1)^n$ as χ ranges on the set of characters of $k^{r\times}$. As a consequence, we show that all non-trivial relations among these sums can be expressed as combinations of Frobenius invariance and the Hasse-Davenport product formula.

[1] Rojas-León A., *Equidistribution and independence of Gauss sums*, arXiv:2207.12439.

The generalized Zeta functions of a linear recurrence sequence

Álvaro Serrano Holgado

Universidad de Salamanca
email: adolfo.quiros@uam.es

Abstract

Given a Perron number α with minimal polynomial $P(x)$ and a sequence of positive integers $\{a_n\}$ satisfying the linear recurrence determined by $P(x)$, one can define the Dirichlet series associated to $\{a_n\}$,$\sum a_n^{-s}$. In this talk I will show how this Dirichlet series, whose half-plane of convergence is the region $\{\mathfrak{Re}(s) > 0\}$, has an analytic continuation to a meromorphic function $\varphi(s)$ of the whole plane, determining in the process its pole set and residues. It turns out that the pole set depends only on α, not on the particular sequence $\{a_n\}$ chosen, that α can be recovered from the residue at $s = 0$ of $\varphi(s)$, and that some properties of the zeta function $\varphi(s)$ are related to the Diophantine properties of α.

I will also show how, from this analysis of the function $\varphi(s)$ and using a generalisation of a formula of Ramanujan, similar results about analytic continuation can be proved for the Hurwitz-type and Lerch- type Dirichlet series $\sum(a_n + x)^{-s}$ and $\sum z^n(a_n + x)^{-s}$ associated to the sequence $\{a_n\}$.

[1] Serrano Holgado Á., *The Tribonacci Dirichlet series*, Acta Mathematica Hungarica 170 (2023), p. 102–109.

[2] Navas L.M., Serrano Holgado Á., *The zeta function of a recurrence sequence of arbitrary degree*, Mediterranean Journal of Mthematics 20 (2023).

[3] Navas L.M., Serrano Holgado Á., *The Lerch-type zeta function of a recurrence sequence of arbitrary degree*, arXiv:2303.16602 (2023).

Frobenius structure and p-adic zeta function

Masha Vlasenko

IMPAN
Warszawa
email: mvlasenko@impan.pl

Abstract

I will explain how differential operators coming from algebraic geometry produce interesting p-adic numbers. In a recent work with Frits Beukers we give examples of families of Calabi-Yau hypersurfaces in n dimensions, for which one observes p-adic zeta values $\zeta_p(k)$ for $1 < k < n$. Appearance of p-adic zeta values for differential operators of Calabi-Yau type was conjectured by Candelas, de la Ossa and van Straten.